Exam 2 review

Tuesday, October 22, 2024 4:15 PM

Exam_2

Math 241

March 11, 2024

Exam 2 - Time: 50 minutes

Dr. Ebrahimian

Please clearly print your name and UID in the appropriate space.

Name:

UID:

- Calculators, formula sheets, notes, and electronic devices, are not allowed. Only writing utensils, sharpeners and erasers are allowed.
- Show your work completely and clearly.
- Answer each problem on the allocated space only.
- Please **turn off all electronic devices**. No electronic device may be on or visible at any time during the exam.
- There are a total of 101 points to be earned. One extra point thanks to the primeness of 101!
- Unless you are asked to, you do not need to simplify your final numerical answers.
- Unless specified, you need to evaluate all integrals.
- Quantities such as $\ln 1$, $\sin(\pi/4)$, etc. must be replaced by their numerical values.
- **Please do not start before you are told to do so.**
- All pages are double sided.

1a. (15 pts) Approximate $\sqrt{(2.01)^3 + (0.98)^3}$ using the tangent plane approximation. Simplify your answer.

do it in class

1b. (10 pts) Find the direction, as a unit vector, in which the function $x^2y + y^2e^x$ decreases most rapidly at $\leadsto$ $(0, 1)$. What is this minimum directional derivative?

Th: • $D_u f$ maximizes at $u = \nabla f$ and max $= \|\nabla f\|$

• $D_u f$ minimizes at $u = -\nabla f$ and min $= -\|\nabla f\|$

Def: $D_u f = \nabla f \cdot \frac{u}{\|u\|}$

A: $u' = -\nabla f$

$f(x,y) = x^2y + y^2e^x$

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 2xy + y^2e^x \\ x^2 + 2ye^x \end{bmatrix}_{(0,1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$u' = -\nabla f(0,1) = -(1,2) \leadsto u = \frac{u'}{\|u'\|} = \frac{-(1,2)}{\|(1,2)\|} = -\frac{1}{\sqrt{5}} \cdot (1,2)$

The min value is $D_u f = -\|\nabla f\| = -\|(1,2)\| = -\sqrt{5}$

Alternatively $= \nabla f \cdot \frac{u}{\|u\|} = \nabla f \cdot \frac{-\nabla f}{\|\nabla f\|} = -\frac{\nabla f \cdot \nabla f}{\|\nabla f\|} = -\frac{\|\nabla f\|^2}{\|\nabla f\|} = \boxed{-\|\nabla f\|}$

$= -\|(1,2)\|$

$= -\sqrt{5}$

2. Consider the function $f(x,y) = \dfrac{y^6}{x^4+y^6}$ $x^2+y^2=r^2$

2a. (13 pts) Evaluate the limit or show it does not exist: $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

Paths: (i) $y=0$, $x\to 0$ $\Big\}$ $\lim\limits_{(x,y)\to(0,0)} \dfrac{0}{x^4+0} = \lim \dfrac{0}{x^4} = \lim 0 = 0$

$\not\parallel$ so limit DNE

(ii) $x=0$, $y\to 0$: $\lim\limits_{y\to 0} \dfrac{y^6}{0+y^6} = 1$

- (iii) $x=y$ $\quad x=my$ $\rightsquigarrow \lim\limits_{x\to 0} \dfrac{x^6}{x^4+x^6} = \lim\limits_{x\to 0} \dfrac{x^6}{x^4(1+x^2)} = \lim\limits_{x\to 0} \dfrac{x^2}{1+x^2} = 0$

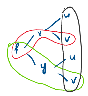
- (iv) $x=y^2$ $\quad x=y^a$

(v) $y=x^2$

2b. (12 pts) Given $x = u + v^2$, and $y = u - v$, use Chain Rule to find $\dfrac{\partial f}{\partial v}$. Your answer may be in terms of u, v, x, y. You **must** use the Chain Rule for this problem.

$f(x,y) = \dfrac{y^6}{x^4+y^6}$ $\quad x = u+v^2$ $\quad y = u-v$

$\dfrac{\partial f}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v}$ $\qquad \left(\dfrac{4^6}{x^4+4^6}\right)$



$f = \dfrac{y^6}{x^4+y^6} \rightsquigarrow f_x = \dfrac{\cancel{(y^6)'\cdot(x^4+y^6)} - y^6\cdot(x^4+y^6)'}{(x^4+y^6)^2} = \dfrac{-y^6\cdot 4x^3}{(x^4+y^6)^2}$ $\quad 0$

$f_y = \dfrac{(y^6)'\cdot(x^4+y^6) - y^6\cdot(x^4+y^6)'}{(x^4+y^6)^2} = \dfrac{6y^5\cdot(x^4+y^6) - y^6\cdot 6y^5}{(x^4+y^6)^2}$

$\dfrac{\partial f}{\partial v} = f_x\cdot\dfrac{\partial x}{\partial v} + f_y\dfrac{\partial y}{\partial v}$

$= \dfrac{-y^6 4x^3}{(x^4+y^6)^2}\cdot(2v) + \left(\dfrac{6y^5(x^4+y^6)-6y^{11}}{(x^4+y^6)^2}\right)\cdot(-1)$

Comment: (plug $x = u+v^2$, $y = u-v$ and that is your final answer)

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3. Consider the function $f(x,y) = x^2 - 4xy + y^3$.

3a. (11 pts) Determine all critical points of $f(x,y)$.

$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 2x-4y \\ -4x+3y^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 2x-4y=0 \rightsquigarrow \boxed{x=2y} \\ -4x+3y^2=0 \rightsquigarrow -4\cdot(2y)+3y^2=0 \end{cases}$

$-8y+3y^2=0 \Rightarrow y(-8+3y)=0 \Rightarrow y=0, \frac{8}{3}$

- $y=0$: $x=2y=0 \rightsquigarrow P_1=(0,0)$
- $y=\frac{8}{3}$: $x=2y=2\cdot\frac{8}{3}=\frac{16}{3} \rightsquigarrow P_2=\left(\frac{16}{3}, \frac{8}{3}\right)$

$\rightsquigarrow H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -4 & 6y \end{bmatrix} \rightsquigarrow \det(H)$

- <0 saddle pt
- >0: $f_{xx}>0 \rightsquigarrow$ local min; $f_{xx}<0 \rightsquigarrow$ local max
- $=0$ indeterminate

3b. (15 pts) Determine if each critical point is a local maximum, a local minimum or a saddle point.

- For $P_1=(0,0)$: $H=\begin{bmatrix} 2 & -4 \\ -4 & 0 \end{bmatrix} \rightsquigarrow \det = 2\cdot 0-16 = -16 < 0 \rightsquigarrow$ saddle pt
- For $P_2=\left(\frac{16}{3}, \frac{8}{3}\right)$: $H=\begin{bmatrix} 2 & -4 \\ -4 & 6\cdot\frac{8}{3} \end{bmatrix} \rightsquigarrow \det = 2\cdot 16-16=16>0$, $f_{xx}=2>0$ $\Big\} \Rightarrow$ local min.

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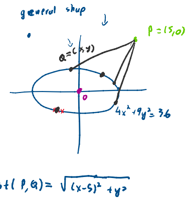
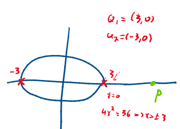
4. (25 pts) Using Lagrange Multiplier's method find the minimum and maximum distance from the point $(5,0)$ to the points on the ellipse $4x^2+9y^2=36$. Assume these minimum and maximum values exist.

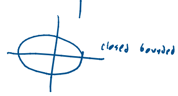
general shup



Actual shape



$\text{dist}(P,Q) = \sqrt{(x-5)^2+y^2}$



Goal : Minimize/maximize the distance

So $\boxed{f(x,y) = \sqrt{(x-5)^2+y^2}}$

$\boxed{\text{Constraint}: 4x^2+9y^2=36}$

To simplify calculations, you can replace f by $h = f^2 = (x-5)^2+y^2$

$h(x,y) = (x-5)^2+y^2$

Constraint: $\underbrace{4x^2+9y^2}_{g(x,y)}=36$

$g(x,y) = 4x^2+9y^2$

$\nabla f = \lambda \nabla g = \begin{bmatrix} 2(x-5) \\ 2y \end{bmatrix} = \lambda\cdot\begin{bmatrix} 8x \\ 18y \end{bmatrix} = \boxed{\begin{bmatrix} 8\lambda x \\ 18\lambda y \end{bmatrix}}$

$\boxed{\text{Lagrange} \begin{cases} 2(x-5) = 8\lambda x \\ 2y = 18\lambda y \\ 4x^2+9y^2=36 \end{cases}}$ $\rightsquigarrow 2y-18\lambda y=0 \Rightarrow 2y(1-9\lambda)=0 \Rightarrow \boxed{y=0}$ or $\boxed{\lambda=\frac{1}{9}}$

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Case 1: $y=0$: $4x^2+9\cdot0^2=36 \Rightarrow 4x^2=36 \Rightarrow x=\pm3 \rightsquigarrow \boxed{Q_1=(3,0)}$, $\boxed{Q_2=(-3,0)}$

Case 2: $\lambda=\frac{1}{9}$: $2(x-5)=\frac{8}{9}x \Rightarrow 2x-10=\frac{8}{9}x \Rightarrow 2x-\frac{8}{9}x=10$

$\Rightarrow \frac{10}{9}x=10 \Rightarrow x=9$

$4\cdot9^2+9y^2=36 \Rightarrow 9y^2=36-4\cdot81<0 \rightsquigarrow$ no solutions so no points

$f=(x-5)^2+y^2$

$f(Q_1)=(3-5)^2+0^2=4$ global min. closest pt is Q_1

$f(Q_2)=(-3-5)^2+0^2=64$ global max. furthest pt is Q_2

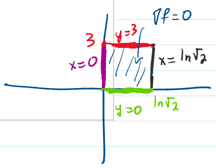
$dist(P,Q_1)=\sqrt{f(Q_1)}=2$ closest distance

$dist(P,Q_2)=\sqrt{f(Q_2)}=8$ furthest distance

Q6] $f(x,y)=ye^{-2x}$

R = rect. region with vertices $(0,0)$, $(\ln\sqrt{2},0)$, $(\ln\sqrt{2},3)$, $(0,3)$ with the interior

$\ln\sqrt{2}=\ln 2^{\frac{1}{2}}=\frac{1}{2}\ln(2)$



Interior: $\nabla f=\begin{bmatrix} f_x \\ f_y \end{bmatrix}=\begin{bmatrix} -2ye^{-2x} \\ e^{-2x} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ no solutions

$e^{-2x} \neq 0$ no crit pts on interior!

Boundary: (1) $x=0$: $f(x,y)=f(0,y)=y\cdot e^0=y$ $0\le y\le3$

max = 3

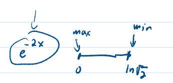
min = 0

$3e^{-2\ln\sqrt{2}}=3e^{-\frac{2}{2}\ln2}=3e^{-\ln2}=\frac{3}{e^{\ln2}}=\frac{3}{2}$

(2) $y=3$: $f(x,y)=f(x,3)=3e^{-2x}$ $0\le x\le\ln\sqrt{2}$

min $=3e^{-2\ln\sqrt{2}}=\frac{3}{2}$

max $=3e^0=3$



(3) $x=\ln\sqrt{2}$: $f(x,y)=f(\frac{1}{2}\ln2,y)=ye^{-2\ln\sqrt{2}}=\frac{y}{2}$ $0\le y\le3$

max = 3/2

min = 0

(4) $y=0$: $f(x,0)=0$ max = min = 0

global max = 3

global min = 0

Problem set ch 13...

10/24 Problem Set

NAME: __________

UID: __________

Problem 1: Find the critical points of the following functions, and determine whether it is a maximum, minimum or a saddle point:

(a) $f(x,y)=e^{xy}$

(b) $f(x,y)=x^3-6x^2-3y^2$

(c) $f(x,y)=e^x(\sin y-1)$

Problem 2: Find three positive numbers x,y,z whose sum is 48 and whose product is as large as possible. How would the question change if you had to compute the smallest possible product? Calculate $x^2+y^2+z^2$ in both the cases.

Problem 3: A rectangular box without top has a volume of 72 cubic meters. Find the dimensions of such a box having the smallest possible surface area. (Remark: for exams, it is a good idea to remember volume and surface area formulae for the standard shapes like cylinder, sphere, parallelopiped).

Problem 4: Use the method of Lagrange Multipliers to find the extreme values of the following functions f (show all the steps)

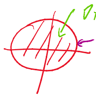
(a) $f(x,y)=x+y^2, x^2+y^2=4$

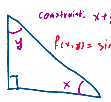
(b) $f(x,y)=x^2+2y^2, x^2+y^2=1$

Problem 5: Find the point on the sphere $x^2+y^2+z^2=1$ that are closest to or farthest from the point $(4,2,1)$.

Problem 6: Let x,y denote the acute angles of a right angled triangle. Then find the maximum value of $\sin x\sin y$.

Problem 7: Let $f(x,y)=3x^2+2y^2-4y+1$, then compute the extreme values of f on the disk $x^2+y^2\le16$. Straight forward



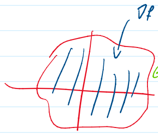


constraint: $x+y=\frac{\pi}{2}$

$f(x,y)=\sin(x)\sin(y)$ maximize

the rest is Lagrange

$f(x,y)=16-x^2-4y^2$ $x^4+2y^4\le1$



Study theory

Problem set ch 13

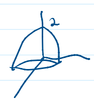
13.1: 1) $x^2+y^2+z^2=1$ sphere

2) $x^2+y^2=1$ cylinder

3) $z^2=x^2+y^2$ cone

4) $z=x^2+y^2$ paraboloid

$z=2-x^2-y^2$



13.1: Level sets

Express the level curve in level set notation and maybe graph it

Ex: $f(x,y)=|x|+|y|$

1) Find level set at $c=1$ and draw it

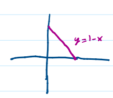
2) Find all other level sets

Level set $=\{(x,y): f(x,y)=c\}$

$\boxed{=\{(x,y): |x|+|y|=1\}}$

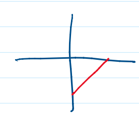
(1) If $x\ge0$ and $y\ge0$: $x+y=1$

$y=1-x$



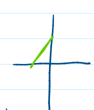
(2) If $x\ge0$ and $y\le0$: $x-y=1$

$y=x-1$

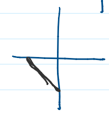


(3) If $x\le0$ and $y\ge0$: $-x+y=1$

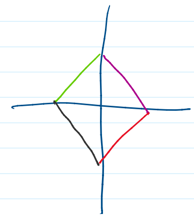
$y=x+1$



(4) If $x\le0$ and $y\le0$



$|x|+|y|=1$



Problem Set

Problem 1. Determine if there's a value of c such that $f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} & (x,y) \neq 0 \\ c & (x,y) = 0 \end{cases}$ is continuous. What if $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq 0 \\ c & (x,y) = 0 \end{cases}$

Problem 2. $f(x,y) = x^3 e^{2xy}$. Compute $f_{xy}(1,2) + 2f_{yx}(1,2)$

Problem 3. $f(x,y) = \sin(x^2y) + x^3 + y^2 + \frac{x}{y}$ Compute ∇f

Problem 4. Let $w = \frac{e^z}{x^2+y^2}, x = u^2+v^2, y = u^2 - v^2, z = uv$. Find $\frac{\partial w}{\partial u}$

Problem 5. Two cars approach an intersection. The first is moving 20m/h the other 100m/h. At what rate is the distance between the two cars changing when the first car is 0.5 miles away from the intersection and the second is 1.2 miles away?

Problem 6. $f(x,y) = (y-x)^{-1}$. Find two unit vectors u, v such that $D_u f(1,2)$ is as large as possible and $D_v(f)(1,2) = 0$

Problem 7. $f(x,y,z) = 3x^2\cos(yz)$. Find the directional derivative of $f(x,y,z)$ at $(1,0,0)$ in the direction of $u = (2,1,-2)$.

13.6. **Problem 8.** $f(x,y) = xy - x + y$. Find an equation of the plane tangent to the graph of f at $P = (0,2)$.

$g(x,y,z) = z - f(x,y) = z - xy + x - y$

Problem 9. $\sin(xy) = 2 - z^2$. Find an equation of the plane tangent to the surface at $P = (\pi, \frac{1}{2}, -1)$

Problem 10. Approximate $\sqrt{(2.01)^3 + (0.98)^3}$ using the tangent plane approximation. Simplify your answer.

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}} \qquad \boxed{\lim_{(x,y)} \frac{x^3+y^3}{x^2+y^2}}$$

$$\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} 6x\cos(yz) \\ -3x^2 z\sin(yz) \\ 3x^2 y\sin(yz) \end{bmatrix}_{(1,0,0)} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$D_u f = \nabla f \cdot \frac{u}{\|u\|} = (6,0,0) \cdot \frac{(2,1,-2)}{\|(2,1,-2)\|} = \frac{12}{\|(2,1,-2)\|} = \frac{12}{\sqrt{9}} = \frac{12}{3} = 4$$

1) Compute $D_u f \rightsquigarrow \nabla f \cdot \frac{u}{\|u\|}$

2) At what direction $D_u f$ maximizes/minimizes $\rightsquigarrow$ $u = \pm \nabla f$ $\rightsquigarrow u' = \frac{u}{\|u\|}$

or what is the max/min $\rightsquigarrow$ max/min $= \pm \|\nabla f\|$

9) $\sin(xy) = 2 - z^2 \rightsquigarrow \sin(xy) + z^2 = 2$ $\qquad f(x,y,z) = \sin(xy) + z^2$

Level set $f = 2 \leftarrow$

$$\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} y\cos(xy) \\ x\cos(xy) \\ 2z \end{bmatrix}_{(\pi, \frac{1}{2}, -1)} = \begin{bmatrix} \frac{1}{2}\cos(\frac{\pi}{2}) \\ \pi\cos(\frac{\pi}{2}) \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} = N = \text{normal vector}$$

$E: -2(z+1) = 0$

Limit Problems:

$$f(x,y) = \frac{x^2y}{x^2+y^2}$$

$x^2 + y^2 = r^2$

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^2(\theta)\sin(\theta)}{r^2}$$

$$= \lim_{r\to 0} r\cos^2(\theta)\sin(\theta) \overset{?}{=} 0$$

$\cos(\theta) \leq 1$

$\sin(\theta) \leq 1$

$$\underset{\substack{\downarrow r\to 0 \\ 0}}{0} \leq \underbrace{\left| r\cos^2(\theta)\sin(\theta) \right|}_{\substack{\downarrow \text{squeeze} \\ 0}} \leq |r \cdot 1^2 \cdot 1| = \underset{\substack{\downarrow r\to 0 \\ 0}}{|r|}$$